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Consistency and Rate of Convergence of Switched Least Squares System Identification for Autonomous Markov Jump Linear Systems

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***Abstract*— In this paper, we investigate the problem of system identification for autonomous Markov jump linear systems (MJS) with complete state observations. We propose switched least squares method for identification of MJS, show that this method is strongly consistent, and derive data-dependent and data-independent rates of convergence. In particular, our data-dependent rate of convergence shows that, almost surely, the system identification error is $\mathcal{O}\big(\sqrt{\log(T)/T}\big)$ where T is the time horizon. These results show that switched least squares method for MJS has the same rate of convergence as least squares method for autonomous linear systems. We compare our results with those in the literature and present numerical examples to illustrate the performance of the proposed system identification method.**

I. Introduction

Markov jump linear systems (MJS) are a good approximation of non-linear time-varying systems arising in various applications including networked control systems [1] and cyber-physical systems [2], [3]. There is a rich literature on the stability analysis (e.g., [4], [5], [6]) and optimal control (e.g., [7]) of MJS. However, most of the literature assumes that the system model is known. The problem of system identification, i.e., identifying the dynamics from data, has not received much attention in this setup.

The problem of identifying the system model from data is a key component for control synthesis for both offline control methods and online control methods including adaptive control and reinforcement learning [8]. A commonly used method for system identification is the least squares method. Asymptotic rates of convergence and strong consistency of least squares method for regression are provided in [9]. These results are extended to autonomous linear systems by [10] and ARMAX systems in [11], [12], [13]. See Chapter 6 of [14] for a unified overview.

These classical results provide asymptotic convergence guarantees. In recent years, there has been a significant interest in the machine learning community to establish finite-time convergence guarantees for system identification under a variety of assumptions [15], [16], [17], [18], [19], [20], [21], [22].

System identification of MJS and switched linear systems (SLS) has received less attention in the literature. There is some work on designing asymptotically stable controllers for unknown SLS [23], [24], [25] but these papers do not established rates of convergence for system identification. The problem of identification of SLSs using set membership identification has been investigated in [26], [27]. There are few recent results which establish high probability rates of converge for different models of SLS and MJS for subspace methods [28] and least-square methods [29]. We provide a detailed comparison with these papers in Sec. V.

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A. Contribution

We investigate the problem of identifying an unknown (autonomous) MJS. We propose a *switched* least squares method for system identification and provide data-dependent and data-independent rates of convergence for this method. Using these bounds, we establish *strong* consistency of the switched least squares method and establish a $\mathcal{O}(\sqrt{\log(T)/T})$ rate of convergence, which matches with the rate of convergence of non-switched linear systems established in [10]. In contrast to the existing high-probability convergence guarantees in the literature, our results show that the estimation error converges to zero *almost surely*. To the best of our knowledge, this is the first result in the literature which establishes strong consistency and almost sure rates of convergence for MJS.

B. Organization

The rest of the paper is organized as follows. In Sec II, we state the system model, assumptions, and the main results. In Sec. III, we prove the main results. We present an illustrative example in Sec. IV and compare our assumptions and results with the existing literature in Sec. V. Finally, we conclude in Sec. VI.

C. Notation

Given a matrix A, $A(i,j)$ denotes its (i,j)-th element, $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the largest and smallest magnitudes of right eigenvalues, $\sigma_{\max}(A) = \sqrt{\lambda_{\max}(A^\intercal A)}$ denotes the spectral norm. For a square matrix Q, $\mathrm{tr}(Q)$ denotes the trace. When Q is symmetric, $Q \succeq 0$ and $Q \succ 0$ denotes that Q is positive semi-definite and positive definite, respectively. For two square matrices, Q_1 and Q_2 of the same dimension, $Q_1 \succeq Q_2$ means $Q_1 - Q_2 \succeq 0$.

Given a sequence of positive numbers $\{a_t\}_{t\geq 0}, a_T = \mathcal{O}(T)$ means that $\limsup_{T\to\infty} a_T/T < \infty$, and $a_T = o(T)$ means that $\limsup_{T\to\infty} a_T/T = 0$. Given a sequence of vectors $\{x_t\}_{t\in\mathcal{T}}$, $\mathrm{vec}(x_t)_{t\in\mathcal{T}}$ denotes the vector formed by vertically stacking $\{x_t\}_{t\in\mathcal{T}}$. Given a sequence of random variables $\{x_t\}_{t\geq 0}$, $x_{0:t}$ is a short hand for $(x_0, \cdots, x_t)$ and $\sigma(x_{0:t})$ denotes the sigma field generated by random variables $x_{0:t}$.

$\mathbb{R}$ and $\mathbb{N}$ denote the set of real and natural numbers. For a set $\mathcal{T}$, $|\mathcal{T}|$ denotes its cardinality. For a vector x, $\|x\|$ denotes the Euclidean norm. For a matrix A, $\|A\|$ denotes the spectral norm and $\|A\|_\infty$ denotes the element with the largest absolute value. Convergence in almost sure sense is abbreviated as $a.s.$

II. System model and problem formulation

Consider a discrete-time (autonomous) MJS. The state of the system has two components: a discrete component $s_t \in \{1, \dots, k\}$ and a continuous component $x_t \in \mathbb{R}^n$. There is a finite set $\mathcal{A} = \{A_1, \dots, A_k\}$ of system matrices, where $A_i \in \mathbb{R}^{n\times n}$. The continuous component x_t of the state starts at a fixed value x_0 and the initial discrete state s_t starts according to a prior distribution π_0. The continuous state evolves according to:

$$x_{t+1} = A_{s_t} x_t + w_t, \quad t \geq 0, \tag{1}$$

where $\{w_t\}_{t\geq 0}$, $w_t \in \mathbb{R}^n$, is a noise process. The discrete component evolves in a Markovian manner according to a time-homogeneous irreducible and aperiodic transition matrix P, i.e. $\mathbb{P}(s_{t+1} = j | s_t = i) = P_{ij}$. Let $\pi_t = (\pi_t(1), \dots, \pi_t(k))$ denote the distribution of the discrete state at time t and π_∞ denote the stationary distribution. Let $\mathcal{F}_{t-1} = \sigma(x_{0:t}, s_{0:t})$ denote the sigma-algebra generated by the history of the complete state. Furthermore, let σ_i denote the maximum singular value of A_i, $i \in \{1, \dots, k\}$. It is assumed that the model satisfies the following assumptions:

Assumption 1. *The noise process $\{w_t\}_{t\geq 0}$ is a martingale difference sequence with respect to $\{\mathcal{F}_t\}_{\geq 0}$, i.e., $\mathbb{E}[|w_t|] < \infty$ and $\mathbb{E}[w_t \mid \mathcal{F}_{t-1}] = 0$. Furthermore, there exists a constant $\alpha > 2$ such that $\sup_{t\geq 0} \mathbb{E}[\|w_t\|^\alpha \mid \mathcal{F}_{t-1}] < \infty$ a.s. and there exists a symmetric and positive definite matrix $C \in \mathbb{R}^{n\times n}$ such that $\liminf_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} w_t w_t^\intercal = C$ a.s.*

Assumption 2. *The stationary distribution $\pi_\infty = (\pi_\infty(1), \dots, \pi_\infty(k))$ satisfies $\pi_\infty(i) \neq 0$ for all i and $\prod_{i=1}^k \sigma_i^{\pi_\infty(i)} < 1$.*

Assumption 1 is a standard assumption in the asymptotic analysis of system identification of linear systems [14], [9], [10], [11], [12] and allows the noise process to be non-stationary and have heavy tails (as long as moment condition is satisfied).

Assumption 2 is a standard assumption for almost sure exponential stability of *noise-free* switched linear system i.e., when $w_t = 0$ [4]. Some of the recent results on system identification of Markov jump linear systems impose slightly different assumptions and we compare with those in Sec. V.

A. System identification and switched least squares estimates

We are interested in the setting where the system dynamics $\mathcal{A}$ and the switching transition matrix P are unknown. Let $\theta^\intercal = [A_1, \dots, A_k] \in \mathbb{R}^{n\times nk}$ denote the unknown parameters of the system dynamics matrices. We consider an agent that observes the complete state (x_t, s_t) of the system at each time and generates an estimate $\hat{\theta}_T$ of θ as a function of the observation history $(x_{0:T}, s_{0:T})$. A commonly used estimate in such settings is the least squares estimate:

$$\hat{\theta}_T^\intercal = \operatorname*{arg\,min}_{\theta^\intercal = [A_1,\dots,A_k]} \sum_{t=0}^{T-1} \|x_{t+1} - A_{s_t} x_t\|^2. \tag{2}$$

The components $[\hat{A}_{1,T}, \dots, \hat{A}_{k,T}] = \hat{\theta}_T^\intercal$ of the least squares estimate can be computed in a switched manner. Let $\mathcal{T}_{i,T} = \{t \leq T \mid s_t = i\}$ denote the time indices until time T when the discrete state of the system equals i. Note that for each $t \in \mathcal{T}_{i,T}$, $A_{s_t} = A_i$. Therefore, we have

$$\hat{A}_{i,T} = \operatorname*{arg\,min}_{A_i \in \mathbb{R}^{n\times n}} \sum_{t\in\mathcal{T}_{i,T}} \|x_{t+1} - A_i x_t\|^2, \quad \forall i \in \{1, \cdots, k\}. \tag{3}$$

Let $X_{i,T} = \sum_{t\in\mathcal{T}_{i,T}} x_t x_t^\intercal$ denote the unnormalized empirical covariance of the continuous component of the state at time instant T when the discrete component equals i. Then, $\hat{A}_{i,T}$ can be computed recursively as follows:

$$\hat{A}_{i,T+1} = \hat{A}_{i,T} + \left[\frac{X_{i,T}^{-1} x_T (x_{T+1} - \hat{A}_{i,T} x_T)^\intercal}{1 + x_T^\intercal X_{i,T}^{-1} x_T}\right] \mathbb{1}\{s_{T+1} = 1\} \tag{4}$$

where $X_{i,T}$ may be updated as $X_{i,T+1} = X_{i,T} + [x_{T+1} x_{T+1}^\intercal] \mathbb{1}\{s_{T+1} = 1\}$. Due to the switched nature of the least squares estimate, we refer to above estimation procedure as *switched least squares* system identification.

B. The main results

A fundamental property of any sequential parameter estimation method is strong consistency, which we define below.

Definition 1. *An estimator $\hat{\theta}_T$ of parameter θ is called strongly consistent if $\lim_{T\to\infty} \hat{\theta}_T = \theta$, a.s.*

Our main result is to establish that the switched least squares estimator is strongly consistent. We do so by providing two different characterization of the rate of convergence. We first provide a data-dependent rate of convergence which depends on the spectral properties of the unnormalized empirical covariance. We then present a data-independent characterization of rate of convergence which only depends on T. All proofs are presented in Sec. III.

Theorem 1. *Under Assumptions 1 and 2, the switched least squares estimates $\{\hat{A}_{i,T}\}_{i=1}^k$ are strongly consistent, i.e., for each $i \in \{1, \dots, k\}$, we have: $\lim_{T\to\infty} \|\hat{A}_{i,T} - A_i\|_\infty = 0$, a.s.*

Furthermore, the rate of convergence is upper bounded by the following expression:

$$\|\hat{A}_{i,T} - A_i\|_\infty \leq \mathcal{O}\left(\sqrt{\frac{\log[\lambda_{\max}(X_{i,T})]}{\lambda_{\min}(X_{i,T})}}\right), \quad a.s.$$

Remark 1. *Theorem 1 is not a direct consequence of the decoupling procedure in switched least squares method. The k least squares problems have a common covariate process. Therefore, the convergence of the switched least squares method and the stability of the switched linear systems* *are interconnected problems. Our proof techniques leverage this connection to establish the consistency of the system identification method.*

We simplify the result in Theorem 1 and characterize the data dependent result found in Theorem 1 in terms of horizon T and the cardinality of the set $\mathcal{T}_{i,T}$.

Corollary 1. *Under Assumptions 1 and 2, for each $i \in \{1, \ldots, k\}$, we have:*

$$\left\|\hat{A}_{i,T} - A_i\right\|_\infty \leq \mathcal{O}\left(\sqrt{\left(\log(T)\right)/|\mathcal{T}_{i,T}|}\right), \quad \textit{a.s.}$$

Remark 2. *The assumption that $\pi_\infty(i) \neq 0$ implies that for sufficiently large T, $|\mathcal{T}_{i,T}| \neq 0$ almost surely, therefore the expressions in above bounds are well defined.*

The result of Corollary 1 still depends on the data. When system identification results are used for adaptive control or reinforcement learning, it is useful to have a data-independent characterization of the rate of convergence. We present this characterization in the next theorem.

Theorem 2. *Under Assumptions 1 and 2, the rate of convergence of the switched least squares estimator $\hat{A}_{i,T}$ is upper bounded by:*

$$\left\|\hat{A}_{i,T} - A_i\right\|_\infty \leq \mathcal{O}\left(\sqrt{\log(T)/\pi_\infty(i)T}\right), \quad \textit{a.s.}$$

where the constants in the $\mathcal{O}(\cdot)$ notation do not depend on Markov chain $\{s_t\}_{t\geq 0}$ and horizon T. Therefore, the estimation process is strongly consistent, i.e., $\lim_{T\to\infty}\left\|\hat{\theta}_T - \theta\right\|_\infty = 0$ a.s. with the convergence rate given by:

$$\left\|\hat{\theta}_T - \theta\right\|_\infty \leq \mathcal{O}\left(\sqrt{\log(T)/\pi^* T}\right), \quad \textit{a.s.}$$

where $\pi^* = \min_j \pi_\infty(j)$.

Theorem 2 shows that Assumptions 1 and 2 guarantee that the switched least squares method for MJS has the same rate of convergence of $\mathcal{O}(\sqrt{\log(T)/T})$ as non-switched case established in [10]. Moreover, the constants show that the estimation error of i-th least squares problem is proportional to $1/\sqrt{\pi_\infty(i)}$; therefore, the rate of convergence of $\hat{\theta}_t$ is proportional to $1/\sqrt{\pi^*}$, where π^* is the smallest probability in the stationary distribution π_∞.

Remark 3. *SLS is a special case of MJS in which the discrete state evolves in an i.i.d. manner. The presented results in this section are valid for the SLS with substituting stationary distribution π_∞ with the i.i.d. PMF defined over discrete state.*

III. Proof of the Main Results

In this section, we present the proof of Theorems 1 and 2 and Corollary 1. In Section III-B, we review the background on the rate of convergence for least squares regression. In Section III-B, we characterize the asymptotic behaviors of continuous state of the system and covariates of the i-th least squares problem. The proof the of main theorems are presented in Section III-C.

A. Background on least square estimator

Given a filtration $\{\mathcal{G}_t\}_{t\geq 0}$, consider the following regression model:

$$y_t = \beta^\intercal z_t + w_t, \quad t \geq 0, \tag{5}$$

where $\beta \in \mathbb{R}^n$ is an unknown parameter, $z_t \in \mathbb{R}^n$ is $\mathcal{G}_{t-1}$-measurable covariate process, y_t is the observation process, and $w_t \in \mathbb{R}$ is a noise process satisfying Assumption 1 with $\mathcal{F}_t$ replaced by $\mathcal{G}_t$. Then the least squares estimate $\hat{\beta}_T$ of β is given by:

$$\hat{\beta}_T = \arg\min_{\beta^\intercal} \sum_{\tau=0}^{T} \|y_\tau - \beta^\intercal z_\tau\|^2. \tag{6}$$

The following result by [9] characterizes the rate of convergence of $\hat{\beta}_T$ to β in terms of unnormalized covariance matrix of covariates $Z_T \coloneqq \sum_{\tau=0}^{T} z_\tau z_\tau^\intercal$.

Theorem 3 (**Theorem 1 of [9]**). *Suppose the following conditions are satisfied:* **(C1)** *$\lambda_{\min}(Z_T) \to \infty$, a.s. and* **(C2)** *$\log(\lambda_{\max}(Z_T)) = o(\lambda_{\min}(Z_T))$, a.s. Then the least squares estimate in* (6) *is strongly consistent with the rate of convergence:*

$$\|\hat{\beta}_T - \beta\|_\infty = \mathcal{O}\left(\sqrt{\frac{\log\left[\lambda_{\max}(Z_T)\right]}{\lambda_{\min}(Z_T)}}\right) \quad \textit{a.s.}$$

Theorem 3 is valid as long as the covariate process $\{z_t\}_{t\geq 0}$ is $\mathcal{G}_{t-1}$-measurable. For the switched least squares system identification if we take $\mathcal{G}_t$ to be equal to $\mathcal{F}_t$ and verify conditions (C1) and (C2) in Theorem 3, then we can use Theorem 3 to establish the strong consistency and rate of convergence. As mentioned earlier in Remark 1, the empirical covariances are coupled across different components due to the system dynamics. For this reason, establishing (C1) and (C2) is non-trivial. In the next section, we establish properties of the system that enable us to prove (C1) and (C2) for the switched least squares system identification.

B. Asymptotic Behavior of Continuous Component

To simplify the notation, we assume that $x_0 = 0$ which does not entail any loss of generality. Let $\Phi(t-1, \tau+1) = A_{s_{t-1}} \cdots A_{s_{\tau+1}}$ denote the state transition matrix where we follow the convention that $\Phi(t, \tau) = I$, for $t < \tau$. Then we can write the dynamics in Eq. (1) of the continuous component of the state in convolutional form as:

$$x_t = \sum_{\tau=0}^{t-1} \Phi(t-1, \tau+1) w_\tau. \tag{7}$$

In the following lemma, we show that Assumption 2 implies that the sum of norms of the state-transition matrices are uniformly bounded.

Lemma 1 (Uniform Boundedness). *Under Assumption 2, there exists a constant $\bar{\Gamma} < \infty$ such that for all $T > 1$, $\sum_{\tau=0}^{T-1} \|\Phi(T-1, \tau+1)\| \leq \bar{\Gamma}$, a.s.*

See Appendix I for proof.

Next, we characterize the asymptotic behavior of state of the system x_τ and the matrix $X_{i,\tau}$.

Proposition 1. *Under Assumptions 1 and 2, the following hold a.s. for each $i \in \{1, \cdots, k\}$:*
(P1) $\sum_{\tau \in \mathcal{T}_{i,T}} \|x_\tau\|^2 = \mathcal{O}(T)$,
(P2) $\lambda_{\max}(X_{i,T}) = \mathcal{O}(T)$, *and*
(P3) $\liminf_{T\to\infty} \lambda_{\min}(X_{i,T})/|\mathcal{T}_{i,T}| > 0$.

See Appendix II for proof.

Corollary 2. *Proposition 1 implies that the system is stable in the average sense. i.e.*

$$\limsup_{T\to\infty} \frac{1}{T} \sum_{\tau=0}^{T-1} \|x_\tau\|^2 < \infty.$$

See Appendix III for proof.

C. Proof of the Main Results

Using the results established in the previous section, we present a proof of the main results

1) Proof of Theorem 1 : To prove this theorem, we check the sufficient conditions in Theorem 3. First notice that $X_{i,T}$ is $\mathcal{F}_{T-1}$ measurable. Also we have:

(C1) By Proposition 1-(P3), we see that $\lambda_{\min}(X_{i,T}) \to \infty$ a.s.; therefore, (C1) in Theorem 3 is satisfied.

(C2) Proposition 1-(P2) and (P3) imply that there exist positive constants C_1, C_2, such that :

$$\limsup_{T\to\infty} \frac{\log(\lambda_{\max}(X_{i,T}))}{\lambda_{\min}(X_{i,T})} \leq \limsup_{T\to\infty} \frac{\log(C_1) + \log(T)}{C_2|\mathcal{T}_{i,T}|} = 0 \quad \text{a.s.} \tag{8}$$

where the last inequality uses the fact that $\pi_\infty(i) > 0$ implies $|\mathcal{T}_{i,T}| = \mathcal{O}(T)$, a.s. Therefore, the second condition of Theorem 3 is satisfied.

Therefore, by Theorem 3, for each $i \in \{1, \cdots, k\}$, we have:

$$\left\|\hat{A}_{i,T} - A_i\right\|_\infty \leq \mathcal{O}\left(\sqrt{\frac{\log\left[\lambda_{\max}(X_{i,T})\right]}{\lambda_{\min}(X_{i,T})}}\right), \quad \text{a.s.}$$

which proves the claim in Theorem 1.

D. Proof of Corollary 1

Corollary 1 is the direct consequence of Theorem 1. The right hand side of Eq. (8) implies that for each i, the estimation error $\|\hat{A}_{i,T} - A_i\|_\infty$ is upper-bounded by $\mathcal{O}\left(\sqrt{\log(T)/|\mathcal{T}_{i,T}|}\right)$, *a.s.*

E. Proof of Theorem 2

We first establish the strong consistency of the parameter $\hat{\theta}_T$. By Theorem 1 and the fact that $k < \infty$, we get:

$$\left\|\hat{\theta}_T - \theta\right\|_\infty \leq \max_{i\in\{1,\cdots,k\}} \mathcal{O}\left(\sqrt{\frac{\log\left[\lambda_{\max}(X_{i,T})\right]}{\lambda_{\min}(X_{i,T})}}\right), \quad \text{a.s.}$$

Therefore the result follows by applying Theorem 1 to the argmax of above equation. For the second part notice that since $\{s_t\}_{t\geq 0}$ is aperiodic and irreducible Markov chain, by the Ergodic Theorem (Theorem 4.1 in [31]) we know, $\lim_{T\to\infty} |\mathcal{T}_{i,T}|/T = \pi_\infty(i)$, a.s. Now, by Corollary 1, we get:

$$\left\|\hat{A}_{i,T} - A_i\right\|_\infty \leq \mathcal{O}\left(\sqrt{\frac{\log(T)}{|\mathcal{T}_{i,T}|}}\right) = \mathcal{O}\left(\sqrt{\frac{\log T}{\pi_\infty(i) T}}\right) \quad \text{a.s.}$$

which is the claim of Theorem 2.

IV. Numerical Simulation

In this section, we illustrate the result of Theorem 1 via an example. Consider a MJS with $n = 2, k = 2$, $A_1 = \left[\begin{smallmatrix}1.5 & 0\\ 0 & 0.2\end{smallmatrix}\right]$, and $A_2 = \left[\begin{smallmatrix}0.01 & 0.1\\ 0.1 & 0.1\end{smallmatrix}\right]$, transition matrix $P = \left[\begin{smallmatrix}1/12 & 11/12\\ 3/4 & 1/4\end{smallmatrix}\right]$ with $\pi_\infty = [0.45, 0.55]$ and i.i.d. $\{w_t\}_{t\geq 0}$ with $w_t \sim \mathcal{N}(0, I)$. Note that the example satisfies Assumptions 1 and 2, but it is not mean square stable (see the next section). We run the switched least squares for the horizon of $T = 8 \times 10^5$ and repeat the experiment for 20 independent runs. We plot the estimation error $e_{i,T} = \|\hat{A}_{i,t} - A_1\|_\infty$ versus time in Fig. 1. The plot shows that the estimation error is converging almost surely even though the system is not mean square stable.

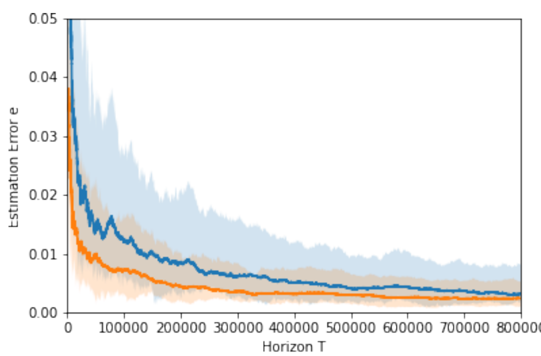

Fig. 1. Performance of switched least squares method for the example of Sec. IV. The solid line shows the mean across 20 runs and the shaded region shows the 25% to 75% quantile bound.

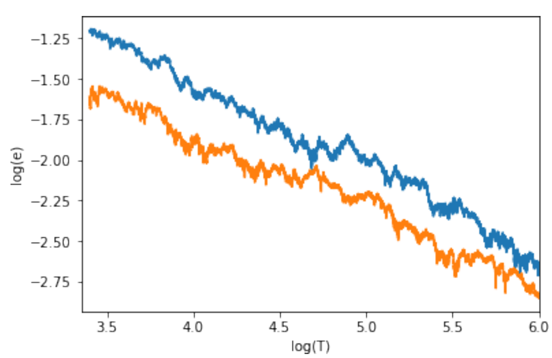

Fig. 2. Logarithm of the estimation error versus logarithm of the horizon is plotted. The linearity of the graph along with approximate slope of -0.5 shows that $e_{i,T} = \tilde{O}(1/\sqrt{T})$.

V. Related Work

As mentioned in the introduction, there are two papers which analyze models similar to ours: [28] and [29]. In this section, we compare our model, assumptions and results with these papers.

The results in [28] investigate the problem of learning the parameters of an unknown SLS of unknown order from input-output data using subspace methods. Under the assumption that the system is mean-square stable, the noise processes are i.i.d. subgaussian, and the system matrices satisfy some technical conditions, they propose an algorithm to estimate an SLS version of the Henkel matrix and obtain parameter estimated by balanced truncation. They show that when the number of samples N_s is sufficiently large, then with high probability the estimation error is $\tilde{\mathcal{O}}(N_s^{-\Delta_s/2})$, where $\Delta_s = \log(1/\rho_{\max})/\log(k/\rho_{\max})$ and $\rho_{\max} = \lambda_{\max}(\sum_{i=1}^{k} p_i A_i \otimes A_i)$.

The results in [28] analyze a subspace-based algorithm, while we analyze a switched least squares algorithm. Both of subspace methods and least squares methods are fundamental methods for system identification of linear systems. The results are derived under different assumptions: we impose a slightly weaker assumption on the noise process and our assumption on the stability of the models are different. Moreover, the nature of the results are different: high probability rates of convergence are provided in [28] while we provide almost sure ones. We discuss the differences between the stability assumptions and the nature of convergence below. Finally, we note that the rate of convergence $\tilde{\mathcal{O}}(N_s^{-\Delta/2})$ depends on the number of subsystems, while our rate of $\tilde{\mathcal{O}}(T^{-1/2})$ does not.

The model analyzed in [29] is an MJS system. Under the assumption that the system is mean square stable, the switching distribution is ergodic and the noise is i.i.d. subgaussian, they propose a system identification procedure where random Gaussian noise is injected as control input and system parameters are estimated using least squares. It is shown in [29] that when T is sufficiently large, then with high probability the estimation error is $\mathcal{O}((\sqrt{k \log T} + \sqrt{\log(1/\delta)})/\sqrt{T})$. Then a certainty equivalent control algorithm is proposed and its regret is analyzed.

The assumptions and the nature of the result in [29] differ in a manner similar to those for [28]. We impose weaker assumptions on the noise process, our assumption on the stability of the models are different, and we provide almost sure rate of convergence. We discuss the difference between the stability assumptions and the nature of the convergence below.

A. Discussion on nature of the convergence

Both [28] and [29] establish high probability rates of convergence. In particular, they show that for any $\delta > 0$ and sufficiently large T, $\|\hat{A}_i - A_i\| \leq \tilde{\mathcal{O}}(f(\delta, T))$ with probability $1 - \delta$, where rate of convergence of $f(\delta, T)$ is $o(T)$ but differs in the two papers. In contrast, our results establish an almost sure rate of convergence. Thus our results imply strong consistency of the system identification while the results of [28] and [29] do not. This is because strong consistency is defined in terms of almost sure convergence, which is a stronger notion of convergence than convergence in probability implied by the high probability bounds.

On the other hand, the results of [28] and [29] are finite-time bounds, i.e., they provide an explicit lower bound on the number of samples needed for the rate of convergence bounds to be valid. In contrast, our result bounds are asymptotic and hold in the limit but do not provide finite time guarantees.

B. Discussion on Stability Assumption

Both [29] and [28] assume that the switched system is mean square stable, i.e., there exist a deterministic vector $x_\infty \in \mathbb{R}^n$ and a deterministic positive definite matrix $Q_\infty \in \mathbb{R}^{n \times n}$ such that for any deterministic initial state $x_0 \in \mathbb{R}$, we have

$$\lim_{\tau \to \infty} \left\| \mathbb{E}[x_\tau] - x_\infty \right\| \to 0 \quad \text{and} \quad \lim_{\tau \to \infty} \left\| \mathbb{E}[x_\tau x_\tau^\mathsf{T}] - Q_\infty \right\| \to 0.$$

As shown in Theorem 3.9 of [6], *mean square stability* is equivalent $\lambda_{\max}(\sum_{i=1}^{k} p_i A_i \otimes A_i) < 1$. Corollary 2 shows that our assumption on stability implies *stability in the average sense* (see [30]), i.e,

$$\limsup_{T \to \infty} \frac{1}{T} \sum_{\tau=0}^{T-1} \|x_\tau\|^2 < \infty.$$

The two notions of stability are different as we illustrate via examples below.

Example 1. *Let $\theta^\mathsf{T} = \{A_1, 0\}$, and $p = (p_1, p_2)$ is an i.i.d. probability transition, with $\lambda_{\max}(p_1 A_1) > 1$ and $x_0 \neq 0$. Then:*

$$\mathbb{E}[x_{\tau+1}] = \mathbb{E}[A_{\sigma_\tau} x_\tau + w_t] = \\ p_1 A_1 \mathbb{E}[x_\tau] = \cdots = (p_1 A_1)^\tau \mathbb{E}(x_0)$$

Which implies:

$$\lim_{\tau \to \infty} \mathbb{E}(x_\tau) = \infty.$$

Therefore, this system is not mean square stable. However, this system satisfies Assumption 2 and therefore is stable in the average sense.

Example 2. *Consider non-switched system with matrix A, with $\lambda_{\max}(A) < 1$ and $\sigma_{\max}(A) > 1$. This system is mean square stable, but it doesn't satisfy Assumption 2.*

VI. Conclusion and Future Directions

In this paper, we investigated the asymptotic performance of the switched least squares for system identification of (autonomous) Markov jump linear systems. We proposed the switched least squares method and established both data dependent and data independent rates of convergence. We showed this method for system identification is strongly consistent and we derived the almost sure rate of convergence of $\mathcal{O}(\sqrt{\log(T)/T})$. This analysis provide a solid first step toward establishing almost sure regret bounds for adaptive control of MJS.

The current results are established for autonomous systems with i.i.d. switching when the complete state of the system is observed. Interesting future research directions include relaxing these modeling assumptions and considering non-autonomous (i.e. controlled) systems under partial observability.

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APPENDIX I
PROOF OF LEMMA 1

Recall that $\sigma_i = \sigma_{\max}(A_i), i \in \{1, \ldots, k\}$. Define $\gamma_t = \sigma_{s_t}$. Then, by sub-multiplicative property of the matrix norms, we have:

$$\begin{aligned} \|\Phi(t-1, \tau+1)\| &= \|A_{s_{t-1}} \ldots A_{s_{\tau+1}}\| \\ &\le \gamma_{t-1} \cdots \gamma_{\tau+1} =: \Gamma_{t-1,\tau+1}. \end{aligned} \tag{9}$$

Given numbers $m_1, \ldots, m_k$, define $f(m_1, \ldots, m_k) = \sigma_1^{m_1} \cdots \sigma_k^{m_k}$. Let $m_i(t-1, \tau+1) = \sum_{t'=\tau+1}^{t-1} \frac{\mathbb{1}\{s_\tau = i\}}{t-\tau-1}$ denote the number of times the discrete state equals i in $[\tau+1, t-1]$. Then,

$$\begin{aligned} \Gamma_{t-1}, \tau+1 &= \gamma_{t-1} \cdots \gamma_{\tau+1} \\ &= f(m_1(t-1,\tau+1), \ldots, m_k(t-1,\tau+1))^{t-\tau-1}. \end{aligned}$$

Since $\{s_t\}_{t\ge 0}$ is aperiodic and irreducible Markov chain, by the Ergodic Theorem (Theorem 4.1 in [31]) we know for any initial distribution π_0, $\lim_{t\to\infty} m_i(t-1,\tau+1) = \pi_\infty(i)$, a.s. Therefore, there exists a $N(\epsilon, \pi_0)$ such that for all $t-\tau-1 \ge N(\epsilon,\pi_0)$, $|m_i(t-1,\tau+1) - \pi_\infty(i)| < \epsilon$ a.s. for all i. Define $N^*(\epsilon) = \sup_{\pi_0 \in \Delta_k} N(\epsilon, \pi_0)$, where Δ_k denotes the k-dimensional simplex. Let π^* denote the corresponding arg sup (which lies in Δ_k due to compactness). Then, $N^* = N(\epsilon, \pi^*)$ is finite due to the Ergodic Theorem. Therefore, for $t-\tau-1 \ge N^*(\epsilon)$, $|m_i(t-1,\tau+1) - \pi_\infty(i)| < \epsilon$.

Furthermore, the rate of convergence of $m_i(t-1,\tau+1)$ to $\pi_\infty(i)$ only depends on $\tau+1$ and $t-1$ only through their difference. By the continuity of $f(\cdot)$, for any $\epsilon' > 0$, there exists a $N'(\epsilon')$ such that for all $t-\tau-1 \ge N'(\epsilon')$, $|f(m_1(t-1,\tau+1), \cdots, m_k(t-1,\tau+1)) - f(\pi_\infty(1), \cdots, \pi_\infty(k))| < \epsilon'$ $a.s.$ Hence, almost surely we have:

$$\begin{aligned} &f(m_1(t-1,\tau+1), \ldots, m_k(t-1,\tau+1)) \\ &\qquad < f(\pi_\infty(1), \ldots, \pi_\infty(k)) + \epsilon' \end{aligned}$$

By Assumption 2, we know $f(\pi_\infty(1),\dots,\pi_\infty(k))<1$. Now we can pick ϵ' such that $f(\pi_\infty(1),\dots,\pi_\infty(k))+\epsilon' =: \beta^* < 1$. Then for all $t\geq 1$,

$$\sum_{\tau=1}^{t-1} f(m_1(t-1,\tau+1),\dots,m_k(t-1,\tau+1))^{t-\tau-1}$$
$$\leq \sum_{\tau=1}^{t-N(\epsilon')-1} \beta^{*t-\tau-1}+$$
$$\sum_{\tau=t-N'(\epsilon')}^{t-1} f(m_1(t-1,\tau+1),\dots,m_k(t-1,\tau+1))^{t-\tau-1}$$
$$<\frac{\beta^{*N'(\epsilon')}}{1-\beta^*}+\sum_{\tau=t-N'(\epsilon')}^{t-1} F_*^{t-\tau-1},$$

where $F_* = \max_{\pi(1),\dots,\pi(k)\in\Delta_k} f(\pi(1),\dots,\pi(k))$, which is clearly bounded. As a result, both terms in the right hand side are bounded which implies the statement in the claim.

APPENDIX II
PROOF OF PROPOSITION 1

We first state the Strong Law of Large Numbers (SLLN) for Martingale Difference Sequences (MDS).

Theorem 4 (Theorem 3.3.1 of [32]). *Suppose $\{X_\tau\}_{\tau\geq 1}$is a martingale difference sequence with respect to the filtration $\{\mathcal{F}_\tau\}_{\tau\geq 1}$. Let a_τ be $\mathcal{F}_{\tau-1}$ measurable and for each $\tau\geq 1$ we have $a_\tau\to\infty$ as $\tau\to\infty$, a.s. If for some $p\in(0,2]$, we have:*

$$\sum_{\tau=0}^{\infty}\frac{\mathbb{E}\big[|X_\tau|^p|\mathcal{F}_{\tau-1}\big]}{a_\tau^p}<\infty,$$

then:

$$\lim_{T\to\infty}\frac{\sum_{\tau=0}^{T}X_\tau}{a_T}=0 \quad a.s.$$

A. Proof of (P1)

We start by the following Lemma which shows the implication of Assumption 1 on the growth rate of energy of the noise process.

Lemma 2 ([10, Eq. (3.1)]). *Under Assumption 1*

$$\sum_{\tau=0}^{T}\|w_\tau\|^2=\mathcal{O}(T), \quad a.s. \tag{10}$$

Using the convolution formula in Eq. (7), we can bound the norm of the state $\|x_t\|^2$ as following:

$$\begin{aligned}\|x_t\|^2 &= \Big(\Big\|\sum_{\tau=0}^{t-1}\Phi(t-1,\tau+1)w(\tau)\Big\|\Big)^2\\ &\overset{(a)}{\leq}\Big(\sum_{\tau=0}^{t-1}\|\Phi(t-1,\tau+1)w(\tau)\|\Big)^2\\ &\overset{(b)}{\leq}\Big(\sum_{\tau=0}^{t-1}\|\Phi(t-1,\tau+1)\|\|w(\tau)\|\Big)^2\\ &\overset{(c)}{\leq}\Big(\sum_{\tau=0}^{t-1}\Gamma_{t,\tau+1}\|w(\tau)\|\Big)^2\end{aligned} \tag{11}$$

where (a) follows from triangle inequality and (b) follow from sub-multiplicative property of the matrix norm, and (c) follows from Eq. (9). Now for a fixed i, $i\in\{1,\cdots,k\}$, we have:

$$\begin{aligned}\sum_{t\in\mathcal{T}_{i,T}}\|x_t\|^2 &\leq \sum_{t\in\mathcal{T}_{i,T}}\Big(\sum_{j=0}^{t-1}\Gamma_{j+1,t-1}\|w(j)\|\Big)^2\\ &\overset{(d)}{\leq}\sum_{t\in\mathcal{T}_{i,T}}\Big(\sum_{j=0}^{t-1}\Gamma_{j+1,t-1}\Big)\Big(\sum_{j=0}^{t-1}\Gamma_{j+1,t-1}\|w(j)\|^2\Big)\\ &\overset{(e)}{\leq}\bar{\Gamma}\sum_{t\in\mathcal{T}_{i,T}}\Big(\sum_{j=0}^{t-1}\Gamma_{j+1,t-1}\|w(j)\|^2\Big)\\ &\overset{(f)}{\leq}\bar{\Gamma}\sum_{j=0}^{T-1}\Big(\sum_{t\in\mathcal{T}_{i,T},j\leq t}\Gamma_{j+1,t-1}\Big)\|w(j)\|^2\\ &\overset{(g)}{\leq}\bar{\Gamma}^2\sum_{j=0}^{T-1}\|w(j)\|^2=\mathcal{O}(T) \quad \text{a.s.}\end{aligned}$$

where (d) follows from Cauchy-Schwarz's inequality, (e) follows from Lemma 1, (f) follows from changing the order of summation, and (g) follows from boundedness of sub-sums of $\sum_{\tau=0}^{T-1}\Gamma_{\tau+1,T-1}$, and Lemma 1.

B. Proof of (P2)

First, notice that we have the following lower and upper bounds for maximum eigenvalue of a matrix:

$$\lambda_{\max}\Big(\sum_{t\in\mathcal{T}_{i,T}}x_tx_t^\intercal\Big)\overset{(a)}{\leq}\text{tr}\Big(\sum_{t\in\mathcal{T}_{i,T}}x_tx_t^\intercal\Big)=\sum_{t\in\mathcal{T}_{i,T}}\|x_i\|^2$$

where (a) follows from the fact that trace of a matrix is sum of its eigenvalues and all eigenvalues of $x_tx_t^\intercal$ are non-negative. Using inequality (a), and Proposition 1-(P1), we get:

$$\lambda_{\max}\Big(\sum_{t\in\mathcal{T}_{i,T}}x_tx_t^\intercal\Big)=\sum_{t\in\mathcal{T}_{i,T}}\|x_i\|^2=\mathcal{O}(T) \quad \text{a.s.}$$

which completes the proof.

C. Proof of (P3)

1) Preliminary Results : First we prove the following preliminary lemma:

Lemma 3. *Assumption 1 and 2 imply:*

$$\sum_{\tau=1}^{\infty}\frac{\|x_\tau\|^2}{\tau^2}<\infty \quad a.s.$$

Proof. The results is a direct consequence of Abel's lemma. Let $S_T := \sum_{\tau=1}^{T}\|x_\tau\|^2$, then we have:

$$\sum_{\tau=1}^{T}\frac{\|x_\tau\|^2}{\tau^2}=\sum_{\tau=1}^{T}\frac{S_\tau-S_{\tau-1}}{\tau^2}=$$
$$\frac{S_T}{T}-\frac{S_0}{1}+\sum_{\tau=2}^{T}S_{\tau-1}\Big(\frac{1}{(\tau-1)^2}-\frac{1}{\tau^2}\Big)$$
$$\overset{(a)}{=}\sum_{\tau=2}^{T}\mathcal{O}\Big(\frac{1}{\tau^2}\Big)<\infty$$

where (a) follows from Proposition 1-(P1), which implies $S_T = \mathcal{O}(T)$. ☐

Lemma 4. *We have the following:*

$$\Big\| \sum_{\tau=1}^{T} A_{s_\tau} x_\tau w_\tau^\intercal + w_\tau x_\tau^\intercal A_{s_\tau}^\intercal \Big\| = o(T) \quad a.s.$$

Proof. We prove the limit element-wise. The (l, p)-th element of the matrix $A_{s_\tau} x_\tau w_\tau^\intercal$ is:

$$\Big[\sum_{j=1}^{n} A_{s_\tau}(l, j) x_\tau(j) \Big] w_\tau(p)$$

Our goal is to prove:

$$\sum_{\tau=1}^{T} \Big(\big[\sum_{j=1}^{n} A_{s_\tau}(l, j) x_\tau(j) \big] w_\tau(p) \Big) = o(T) \quad a.s.$$

In order to show the above expression, we use Theorem 4 and by setting $a_t = t$ and $p = 2$ we show:

$$\sum_{\tau=1}^{T} \frac{\mathbb{E}\Big[\Big(\big[\sum_{j=1}^{n} A_{s_\tau}(l, j) x_\tau(j) \big] w_\tau(p) \Big)^2 \Big| \mathcal{F}_{\tau-1} \Big]}{\tau^2} < \infty \tag{12}$$

We have:

$$\begin{aligned} &\mathbb{E}\Big[\Big(\big[\sum_{j=1}^{n} A_{s_\tau}(l, j) x_\tau(j) \big] w_\tau(p) \Big)^2 \Big| \mathcal{F}_{\tau-1} \Big] \\ &= \sum_{i=1}^{k} P_{di} \mathbb{E}\Big[\Big(\sum_{j=1}^{n} A_i(l, j) x_\tau(j) \Big)^2 w_\tau^2(p) \Big| \mathcal{F}_{\tau-1} \Big] \end{aligned}$$

where in the last expression, it's assumed $s_\tau = d$. Probabilities P_{di} are constant values; therefore, we only prove the boundedness for the inner expectation term. Let $A_* = \max_{i \in \{1,\ldots,k\}} \|A\|_\infty$. Then, for each fixed i, we have:

$$\begin{aligned} &\mathbb{E}\Big[\Big(\sum_{j=1}^{n} A_i(l, j) x_\tau(j) \Big)^2 w_\tau^2(p) \Big| \mathcal{F}_{\tau-1} \Big] \\ &\overset{(a)}{\leq} A_*^2 \sup_\tau \mathbb{E}[w_\tau^2(p) | \mathcal{F}_{\tau-1}] \Big(\sum_{j=1}^{n} x_\tau(j) \Big)^2 \\ &\overset{(b)}{\leq} n A_*^2 \sup_\tau \mathbb{E}\big[w_\tau^2(p) \big| \mathcal{F}_{\tau-1}\big] \sum_{j=1}^{n} x_\tau^2(j) \\ &= n A_*^2 \sup_\tau \mathbb{E}\big[w_\tau^2(p) \big| \mathcal{F}_{\tau-1}\big] \|x_\tau\|^2 \end{aligned}$$

where (a) is because x_τ is $\mathcal{F}_{\tau-1}$ measurable, and (b) is by Cauchy-Schwarz's inequality. Based on Assumption 1, $\mathbb{E}\big[w_\tau^2(p) \big| \mathcal{F}_{\tau-1}\big]$ is uniformly bounded. Therefore the left hand side of Eq. (12) is bounded by:

$$n A_*^2 \sup_\tau \Big\{ \mathbb{E}[w_\tau^2(p) | \mathcal{F}_{\tau-1}] \Big\} \sum_{\tau=1}^{T} \frac{\|x_\tau\|^2}{\tau^2} \overset{(c)}{\leq} \infty$$

where (c) follows from Lemma 3 . ☐

2) Proof of Proposition 1-(P3): Finally, we prove the statement in the proposition. We have:

$$\begin{aligned} x_\tau x_\tau^\intercal =& (A_{s_{\tau-1}} x_{\tau-1} + w_{\tau-1})(A_{s_{\tau-1}} x_{\tau-1} + w_{\tau-1})^\intercal \\ =& A_{s_{\tau-1}} x_{\tau-1} x_{\tau-1}^\intercal A_{s_{\tau-1}}^\intercal \\ &+ A_{s_{\tau-1}} x_{\tau-1} w_{\tau-1}^\intercal + w_{\tau-1} x_{\tau-1}^\intercal A_{s_{\tau-1}}^\intercal + w_{\tau-1} w_{\tau-1}^\intercal. \end{aligned}$$

Since $A_{s_{\tau-1}} x_{\tau-1} x_{\tau-1}^\intercal A_{s_{\tau-1}}^\intercal$ is positive semi definite, we have:

$$x_\tau x_\tau^\intercal \succeq A_{s_{\tau-1}} x_{\tau-1} w_{\tau-1}^\intercal + w_{\tau-1} x_{\tau-1}^\intercal A_{s_{\tau-1}}^\intercal + w_{\tau-1} w_{\tau-1}^\intercal,$$

By summing over $\tau \in \mathcal{T}_{i,T}$, we get:

$$\begin{aligned} \sum_{\tau \in \mathcal{T}_{i,T}} x_\tau x_\tau^\intercal \succeq& \sum_{\tau \in \mathcal{T}_{i,T}} w_{\tau-1} w_{\tau-1}^\intercal \\ &+ \sum_{\tau \in \mathcal{T}_{i,T}} \big[A_{s_{\tau-1}} x_{\tau-1} w_{\tau-1}^\intercal + w_{\tau-1} x_{\tau-1}^\intercal A_{s_{\tau-1}}^\intercal \big] \\ &\overset{(a)}{=} \sum_{\tau \in \mathcal{T}_{i,T}} w_{\tau-1} w_{\tau-1}^\intercal + o(T) \quad \text{a.s.} \end{aligned}$$

where (a) follows from Lemma 4. Furthermore, since $|\mathcal{T}_{i,T}| \to T\pi_\infty(i) \neq 0$ $a.s.$, we have:

$$\begin{aligned} &\liminf_{|\mathcal{T}_{i,T}| \to \infty} \frac{\sum_{\tau \in \mathcal{T}_{i,T}} x_\tau x_\tau^\intercal}{|\mathcal{T}_{i,T}|} \succeq \\ &\qquad \liminf_{|\mathcal{T}_{i,T}| \to \infty} \frac{\sum_{\tau \in \mathcal{T}_{i,T}} w_{\tau-1} w_{\tau-1}^\intercal}{|\mathcal{T}_{i,T}|} \overset{(b)}{=} C \succ 0 \quad \text{a.s.} \end{aligned}$$

where (b) holds by Assumption 1. Therefore

$$\liminf_{|\mathcal{T}_{i,T}| \to \infty} \frac{\sum_{\tau \in \mathcal{T}_{i,T}} x_\tau x_\tau^\intercal}{|\mathcal{T}_{i,T}|} \succ 0$$

implies that:

$$\lambda_{\min}\Big(\liminf_{|\mathcal{T}_{i,T}| \to \infty} \frac{\sum_{\tau \in \mathcal{T}_{i,T}} x_\tau x_\tau^\intercal}{|\mathcal{T}_{i,T}|} \Big) > 0, \quad \text{a.s.}$$

which concludes the proof.

APPENDIX III
PROOF OF COROLLARY 2

Using Eq. (11), we have:

$$\sum_{\tau=1}^{T} \|x_\tau\|^2 = \sum_{i=1}^{k} \sum_{\tau \in \mathcal{T}_{i,T}} \|x_\tau\|^2 \overset{(a)}{=} k\mathcal{O}(T) = \mathcal{O}(T) \quad \text{a.s.}$$

where (a) follows from Prop. 1-(P2).